

GLOBAL WELL-POSEDNESS TO THE 3D INCOMPRESSIBLE MHD EQUATIONS WITH A NEW CLASS OF LARGE INITIAL DATA

RENHUI WAN¹

ABSTRACT. We obtain the global well-posedness to the 3D incompressible magneto-hydrodynamics (MHD) equations in Besov space with negative index of regularity. Particularly, we can get the global solutions for a new class of large initial data. As a byproduct, this result improves the corresponding result in [10]. In addition, we also get the global result for this system in $\chi^{-1}(\mathbb{R}^3)$ originally developed in [12]. More precisely, we only assume that the norm of initial data is exactly smaller than the sum of viscosity and diffusivity parameters.

1. INTRODUCTION

We are concerned with the 3D incompressible MHD equations:

$$\begin{cases} \partial_t u + u \cdot \nabla u - \mu_1 \Delta u + \nabla p = B \cdot \nabla B, \\ \partial_t B + u \cdot \nabla B - B \cdot \nabla u - \mu_2 \Delta B = 0, \\ \operatorname{div} u = \operatorname{div} B = 0, \\ u(0, x) = u_0(x), \quad B(0, x) = B_0(x), \end{cases} \quad (1.1)$$

here $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^3$, u, p, B stand for velocity vector, scalar pressure and magnetic vector, respectively, μ_1 and μ_2 are nonnegative viscosity and diffusivity parameters, respectively.

For $\mu_1 > 0$ and $\mu_2 > 0$, the local well-posedness and global existence with small data for (1.1) were obtained by Duvaut and Lions [7] in d dimensional Sobolev space $H^s(\mathbb{R}^d)$, $s \geq d$. Then Sermange and Termam [16] studied the regularity of weak solutions $(u, B) \in L^\infty(0, T; H^1(\mathbb{R}^3))$. And some regularity criteria were established in [21, 22, 23]. For $\mu_1 > 0$ and $\mu_2 = 0$ (so-called non-resistive MHD equations), by the new Kato-Ponce commutator estimate,

$$\|\Lambda^s(u \cdot \nabla B) - u \cdot \nabla \Lambda^s B\|_{L^2(\mathbb{R}^d)} \leq C \|\nabla u\|_{H^s(\mathbb{R}^d)} \|B\|_{H^s(\mathbb{R}^d)}, \quad s > \frac{d}{2}, \quad d = 2, 3,$$

Fefferman et al. [8] proved the low regularity local well-posedness of strong solutions, which was extended to general inhomogeneous Besov space with initial data $(u_0, B_0) \in B_{2,1}^{\frac{d}{2}-1}(\mathbb{R}^d) \times B_{2,1}^{\frac{d}{2}}(\mathbb{R}^d)$ in the recent works [4] and [18]. Furthermore, for the non-resistive version with smooth initial data near some nontrivial steady state, we refer [13, 14, 15, 24] for the related works.

2010 *Mathematics Subject Classification.* 35Q35, 76W05, 76N10.

Key words and phrases. MHD equations, global well-posedness, large initial data.

Due to a new observation that the velocity field plays a more important role than magnetic field. The new regularity criteria only involving the velocity were proved, see [5, 11, 25, 26] and references therein.

One can easily get a new formation of (1.1) by the following:

$$W^+ := u + B, \quad W^- := u - B, \quad \nu_+ = \frac{\mu_1 + \mu_2}{2}, \quad \nu_- = \frac{\mu_1 - \mu_2}{2}$$

with initial data $W_0^\pm(x) := u_0(x) \pm B_0(x)$, that is,

$$\begin{cases} \partial_t W^+ + W^- \cdot \nabla W^+ - \nu_+ \Delta W^+ + \nabla p = \nu_- \Delta W^-, \\ \partial_t W^- + W^+ \cdot \nabla W^- - \nu_+ \Delta W^- + \nabla p = \nu_- \Delta W^+, \\ \operatorname{div} W^+ = \operatorname{div} W^- = 0, \\ W^+(0, x) = W_0^+(x), \quad W^-(0, x) = W_0^-(x). \end{cases} \quad (1.2)$$

Very recently, He et al. [10] obtained the global well-posedness for (1.2) with initial data (u_0, B_0) satisfying:

(i) $\nu_- = 0$ and

$$\nu_+^{-3} \|W_0^-\|_{L^3}^3 \exp\{C\nu_+^{-3} \|W_0^+\|_{L^3}^3\} < \epsilon_0$$

or

$$\nu_+^{-3} \|W_0^+\|_{L^3}^3 \exp\{C\nu_+^{-3} \|W_0^-\|_{L^3}^3\} < \epsilon_0;$$

(ii) $\nu_- \neq 0$ and

$$\left(\nu_+^{-2} \|W_0^-\|_{\dot{H}^{\frac{1}{2}}}^2 + \frac{\nu_-^2}{\nu_+^2} (\nu_+^{-2} \|W_0^+\|_{\dot{H}^{\frac{1}{2}}}^2 + \frac{\nu_-^2}{\nu_+^2}) \right) \exp \left\{ C\nu_+^{-4} (\|W_0^+\|_{\dot{H}^{\frac{1}{2}}}^4 + \nu_-^4) \right\} < \epsilon_0 \quad (1.3)$$

or

$$\left(\nu_+^{-2} \|W_0^+\|_{\dot{H}^{\frac{1}{2}}}^2 + \frac{\nu_-^2}{\nu_+^2} (\nu_+^{-2} \|W_0^-\|_{\dot{H}^{\frac{1}{2}}}^2 + \frac{\nu_-^2}{\nu_+^2}) \right) \exp \left\{ C\nu_+^{-4} (\|W_0^-\|_{\dot{H}^{\frac{1}{2}}}^4 + \nu_-^4) \right\} < \epsilon_0.$$

Here ϵ_0 is a sufficiently small positive constant.

In this paper, we will prove the global well-posedness of (1.1) ($\mu_1 > 0, \mu_2 > 0$) in generalized space, $\dot{B}_{p,r}^{\frac{3}{p}-1}(\mathbb{R}^3)$, by make full use of the harmonic analysis tools. The details can be given as follows:

Theorem 1.1. *Consider (1.1) with initial data $(u_0, B_0) \in \dot{B}_{p,r}^{\frac{3}{p}-1}(\mathbb{R}^3)$, $(p, r) \in (1, \infty) \times [1, \infty)$, satisfying $\operatorname{div} u_0 = \operatorname{div} B_0 = 0$. There exists a constant C and a small constant $\eta > 0$ such that if*

$$\left(\|W_0^-\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}} + \frac{\nu_-}{\nu_+} (\|W_0^+\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}} + \nu_-) \right) \exp \left\{ C\nu_+^{-\frac{2}{1-\epsilon}} (\nu_- + \|W_0^+\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}})^{\frac{2}{1-\epsilon}} \right\} < \eta\nu_+ \quad (1.4)$$

or

$$\left(\|W_0^+\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}} + \frac{\nu_-}{\nu_+} (\|W_0^-\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}} + \nu_-) \right) \exp \left\{ C\nu_+^{-\frac{2}{1-\epsilon}} (\nu_- + \|W_0^-\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}})^{\frac{2}{1-\epsilon}} \right\} < \eta\nu_+, \quad (1.5)$$

where (ϵ, r) satisfies

$$\begin{cases} 0 \leq \epsilon < 1, & \text{if } r = 1; \\ 0 < \epsilon < 1, & \text{if } 1 < r \leq 2; \\ 1 - \frac{2}{r} \leq \epsilon < 1, & \text{if } 2 < r < \infty. \end{cases} \quad (1.6)$$

Then (1.1) admits a unique global solution (u, B) satisfying

$$(u, B) \in \tilde{C}([0, \infty); \dot{B}_{p,r}^{\frac{3}{p}-1}(\mathbb{R}^3)) \cap \tilde{L}^1([0, \infty); \dot{B}_{p,r}^{\frac{3}{p}+1}(\mathbb{R}^3)).$$

If $\nu_- = 0$, i.e., $\mu_1 = \mu_2 = \nu_+$, we have a corollary immediately.

Corollary 1.2. *Consider (1.1) with initial data $(u_0, B_0) \in \dot{B}_{p,r}^{\frac{3}{p}-1}(\mathbb{R}^3)$, $(p, r) \in (1, \infty) \times [1, \infty)$, satisfying $\operatorname{div} u_0 = \operatorname{div} B_0 = 0$. There exists a constant C and a small constant $\eta > 0$ such that if*

$$\|W_0^-\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}} \exp \left\{ C \nu_+^{-\frac{2}{1-\epsilon}} \|W_0^+\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}}^{\frac{2}{1-\epsilon}} \right\} < \eta \nu_+ \quad (1.7)$$

or

$$\|W_0^-\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}} \exp \left\{ C \nu_+^{-\frac{2}{1-\epsilon}} \|W_0^+\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}}^{\frac{2}{1-\epsilon}} \right\} < \eta \nu_+, \quad (1.8)$$

where (ϵ, r) satisfies (1.6). Then (1.1) admits a unique global solution (u, B) satisfying

$$(u, B) \in \tilde{C}([0, \infty); \dot{B}_{p,r}^{\frac{3}{p}-1}(\mathbb{R}^3)) \cap \tilde{L}^1([0, \infty); \dot{B}_{p,r}^{\frac{3}{p}+1}(\mathbb{R}^3)).$$

Remark 1.3. (i) We will construct the global solution with a new class of large initial data. More precisely, assume that ϕ satisfies the condition in Proposition 2.4, let

$$u_0 = (\partial_2 \phi, -\partial_1 \phi, 0), \quad B_0 = 2 \sin^2 \frac{x_3}{2\epsilon} (\partial_2 \phi, -\partial_1 \phi, 0),$$

then $\operatorname{div} u_0 = \operatorname{div} B_0 = 0$ and $\|(u_0, B_0)\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}} \leq \mathfrak{M}$ ($p > 3$), which is independent of ϵ .

Moreover, thanks to Proposition 2.4, there exists a positive constant C_1 and C_2 ,

$$\begin{aligned} \|u_0\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}} &\geq C_1, \quad \|B_0\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}} \geq \frac{C_1}{2}, \\ \|u_0 - B_0\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}} &\leq C_2 \epsilon^{1-\frac{3}{p}}, \end{aligned}$$

which ensures the conditions (1.4) ($\nu_+ \gg \nu_-$) and (1.7) hold. Additionally, the assumption $\nu_+ \gg \nu_-$ is reasonable in astrophysical magnetic phenomena, see Remark 2.3 in [10]. Combining with the above explanations, this class of large data can lead the global well-posedness to (1.1).

(ii) One can easily check that condition (1.4) is equal to (1.3) when $p = r = 2$ and choosing $\epsilon = \frac{1}{2}$. By Bernstein's inequality, we have the following embedding relationship:

$$\dot{H}^{\frac{1}{2}} \hookrightarrow \dot{B}_{p,r}^{\frac{3}{p}-1}, \quad p > 2, r \geq 2.$$

So our result improves the corresponding work under (1.3) in [10]. By the same way, similar improvements can also be obtained under (1.5), (1.7) and (1.8).

We shall point out that the above result can not be extended to $p = \infty$. As a matter of fact, by these works [2] and [19] concerning the well-known Navier-Stokes equations, (1.1) may ill-posedness in this endpoint Besov space.

Next, we consider the space $\chi^{-1}(\mathbb{R}^3)$, which is smaller than $\dot{B}_{\infty,r}^{-1}$ due to Proposition 2.5. It was originally developed in [12] and applied to get the global well-posedness for the Navier-Stokes equations under

$$\|u_0\|_{\chi^{-1}} < \mu.$$

For MHD equations (1.1), similar result holds under

$$\|u_0\|_{\chi^{-1}} + \|B_0\|_{\chi^{-1}} < \min\{\mu, \eta\}, \quad (1.9)$$

see [20] for details.

We have some new result in $\chi^{-1}(\mathbb{R}^3)$.

Theorem 1.4. *Consider (1.1) with initial data $(u_0, B_0) \in \chi^{-1}(\mathbb{R}^3)$ satisfying $\operatorname{div} u_0 = \operatorname{div} B_0 = 0$. There exists a constant C such that if*

$$\left(\|W_0^-\|_{\chi^{-1}} + \frac{C\nu_-}{\nu_+}(\nu_- + \|W_0^+\|_{\chi^{-1}}) \right) \exp \left\{ \frac{C}{\nu_+^2}(\nu_- + \|W_0^+\|_{\chi^{-1}})^2 \right\} < 2\nu_+ \quad (1.10)$$

or

$$\left(\|W_0^+\|_{\chi^{-1}} + \frac{C\nu_-}{\nu_+}(\nu_- + \|W_0^-\|_{\chi^{-1}}) \right) \exp \left\{ \frac{C}{\nu_+^2}(\nu_- + \|W_0^-\|_{\chi^{-1}})^2 \right\} < 2\nu_+. \quad (1.11)$$

Then (1.1) admits a unique global solution (u, B) satisfying

$$(u, B) \in C([0, \infty); \chi^{-1}(\mathbb{R}^3)) \cap L^1([0, \infty); \chi^1(\mathbb{R}^3)).$$

Similarly, we also have a corollary immediately when $\nu_- = 0$.

Corollary 1.5. *Consider (1.1) with initial data $(u_0, B_0) \in \chi^{-1}(\mathbb{R}^3)$ satisfying $\operatorname{div} u_0 = \operatorname{div} B_0 = 0$. There exists a constant C such that if*

$$\|W_0^-\|_{\chi^{-1}} \exp \left\{ \frac{C}{\nu_+^2} \|W_0^+\|_{\chi^{-1}}^2 \right\} < 2\nu_+$$

or

$$\|W_0^+\|_{\chi^{-1}} \exp \left\{ \frac{C}{\nu_+^2} \|W_0^-\|_{\chi^{-1}}^2 \right\} < 2\nu_+.$$

Then (1.1) admits a unique global solution (u, B) satisfying

$$(u, B) \in C([0, \infty); \chi^{-1}(\mathbb{R}^3)) \cap L^1([0, \infty); \chi^1(\mathbb{R}^3)).$$

Remark 1.6. *The authors in [12] proved the global well-posedness for Navier-Stokes equations by using*

$$\|u \cdot \nabla u\|_{\chi^{-1}} \leq \|u\|_{\chi^{-1}} \|u\|_{\chi^1},$$

while we shall use the new estimate below in our proof, i.e.,

$$\|u \cdot \nabla v\|_{\chi^{-1}} \leq \|u\|_{\chi^0} \|v\|_{\chi^0}.$$

Remark 1.7. *Due to the symmetric structure of (1.2), we only give the proof of Theorem 1.1 and Theorem 1.4 under (1.4) and (1.10), respectively.*

The present paper is structured as follows:

In section 2, we provide some definitions of spaces, establish several lemmas. The third section proves Theorem 1.1, while the last section gives the proof of Theorem 1.4.

Let us complete this section by describing the notations we shall use in this paper.

Notations The uniform constant C is different on different lines. We also use L^p , $\dot{B}_{p,r}^s$ and χ^s to stand for $L^p(\mathbb{R}^d)$, $\dot{B}_{p,r}^s(\mathbb{R}^d)$ and $\chi^s(\mathbb{R}^d)$ in somewhere, respectively. We use $A := B$ to stands for A is defined by B , and $\mathbf{1}$ is the characteristic function.

2. PRELIMINARIES

In this section, we give some necessary definitions, propositions and lemmas.

The Fourier transform is given by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx.$$

Let $\mathfrak{B} = \{\xi \in \mathbb{R}^d, |\xi| \leq \frac{4}{3}\}$ and $\mathfrak{C} = \{\xi \in \mathbb{R}^d, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$. Choose two nonnegative smooth radial function χ , φ supported, respectively, in \mathfrak{B} and \mathfrak{C} such that

$$\sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^d \setminus \{0\}.$$

We denote $\varphi_j = \varphi(2^{-j}\xi)$, $h = \mathfrak{F}^{-1}\varphi$ and $\tilde{h} = \mathfrak{F}^{-1}\chi$, where \mathfrak{F}^{-1} stands for the inverse Fourier transform. Then the dyadic blocks Δ_j and S_j can be defined as follows

$$\Delta_j f = \varphi(2^{-j}D)f = 2^{jd} \int_{\mathbb{R}^d} h(2^j y) f(x-y) dy, \quad S_j f = \sum_{k \leq j-1} \Delta_k f$$

Formally, Δ_j is a frequency projection to annulus $\{\xi : C_1 2^j \leq |\xi| \leq C_2 2^j\}$, and S_j is a frequency projection to the ball $\{\xi : |\xi| \leq C 2^j\}$. One easily verifies that with our choice of φ

$$\Delta_j \Delta_k f = 0 \text{ if } |j - k| \geq 2 \text{ and } \Delta_j (S_{k-1} f \Delta_k f) = 0 \text{ if } |j - k| \geq 5.$$

Let us recall the definition of the Besov space.

Definition 2.1. Let $s \in \mathbb{R}$, $(p, q) \in [1, \infty]^2$, the homogeneous Besov space $\dot{B}_{p,q}^s(\mathbb{R}^d)$ is defined by

$$\dot{B}_{p,q}^s(\mathbb{R}^d) = \{f \in \mathfrak{S}'(\mathbb{R}^d); \|f\|_{\dot{B}_{p,q}^s(\mathbb{R}^d)} < \infty\},$$

where

$$\|f\|_{\dot{B}_{p,q}^s(\mathbb{R}^d)} = \begin{cases} \left(\sum_{j \in \mathbb{Z}} 2^{sqj} \|\Delta_j f\|_{L^p(\mathbb{R}^d)}^q \right)^{\frac{1}{q}}, & \text{for } 1 \leq q < \infty, \\ \sup_{j \in \mathbb{Z}} 2^{sqj} \|\Delta_j f\|_{L^p(\mathbb{R}^d)}, & \text{for } q = \infty, \end{cases}$$

and $\mathfrak{S}'(\mathbb{R}^d)$ denotes the dual space of $\mathfrak{S}(\mathbb{R}^d) = \{f \in \mathcal{S}(\mathbb{R}^d); \partial^\alpha \hat{f}(0) = 0; \forall \alpha \in \mathbb{N}^d \text{ multi-index}\}$ and can be identified by the quotient space of \mathcal{S}'/\mathcal{P} with the polynomials space \mathcal{P} .

The norm of the space $\tilde{L}_t^{r_1}(\dot{B}_{p,r}^s)$ and $\tilde{L}_{t,\omega}^{r_1}(\dot{B}_{p,r}^s)$ is defined by

$$\|f\|_{\tilde{L}_t^{r_1}(\dot{B}_{p,r}^s)} := \|2^{js}\|\Delta_j f\|_{L_t^{r_1} L^p} \|l^r(\mathbb{Z})\|$$

and

$$\|f\|_{\tilde{L}_{t,\omega}^{r_1}(\dot{B}_{p,r}^s)} := \|2^{js}\left(\int_0^t \omega(\tau)^{r_1} \|\Delta_j f(\tau)\|_{L^p}^{r_1} d\tau\right)^{\frac{1}{r_1}}\|l^r(\mathbb{Z})\|.$$

$f \in \tilde{C}(0, t; \dot{B}_{p,r}^s)$ means $f \in \tilde{L}_t^\infty(\dot{B}_{p,r}^s)$ and $\|f(t)\|_{\dot{B}_{p,r}^s}$ is continuous in time.

The following proposition provide Bernstein type inequalities.

Proposition 2.2. *Let $1 \leq p \leq q \leq \infty$. Then for any $\beta, \gamma \in (\mathbb{N} \cup \{0\})^3$, there exists a constant C independent of f, j such that*

1) *If f satisfies*

$$\text{supp } \hat{f} \subset \{\xi \in \mathbb{R}^d : |\xi| \leq \mathcal{K}2^j\},$$

then

$$\|\partial^\gamma f\|_{L^q(\mathbb{R}^d)} \leq C 2^{j|\gamma| + jd(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)}.$$

2) *If f satisfies*

$$\text{supp } \hat{f} \subset \{\xi \in \mathbb{R}^d : \mathcal{K}_1 2^j \leq |\xi| \leq \mathcal{K}_2 2^j\}$$

then

$$\|f\|_{L^p(\mathbb{R}^d)} \leq C 2^{-j|\gamma|} \sup_{|\beta|=|\gamma|} \|\partial^\beta f\|_{L^p(\mathbb{R}^d)}.$$

For more details about Besov space such as some useful embedding relations, see [1, 9, 17].

Lemma 2.3. [6] *Let $1 < p < \infty$, $\text{supp } \hat{u} \subset C(0, R_1, R_2)$ (with $0 < R_1 < R_2$). There exists a constant c depending on $\frac{R_2}{R_1}$ and such that*

$$c \frac{R_1^2}{p^2} \int_{\mathbb{R}^3} |u|^p dx \leq -\frac{1}{p-1} \int_{\mathbb{R}^3} \Delta u |u|^{p-2} u dx. \quad (2.1)$$

Proposition 2.4. *Let $\phi \in \mathcal{S}(\mathbb{R}^3)$, whose Fourier transform supported in annulus contained in $\mathbb{R}^3 \setminus \{0\}$, and $p > 3$. If $u_0 = (\partial_2 \phi, -\partial_1 \phi, 0)$ and $B_0 = 2 \sin^2 \frac{x_3}{2\epsilon} (\partial_2 \phi, -\partial_1 \phi, 0)$, then there exists a constant $C_1, C_2 > 0$ such that*

$$\|u_0\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}} \geq C_1, \quad \|B_0\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}} \geq \frac{C_1}{2}$$

and

$$\|u_0 - B_0\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}} \leq C_2 \epsilon^{1-\frac{3}{p}},$$

here ϵ is sufficiently small.

Proof. The last estimate can be obtained by following the proof of Lemma 3.1 in [3]. So we suffice to show both $\|u_0\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}}$ and $\|B_0\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}}$ has positive lower bound. With this ϕ , there exists a finite $j_0 \in \mathbb{Z}$, such that $\Delta_{j_0} \partial_2 \phi \neq 0$, which implies

$$\|\Delta_{j_0} \partial_2 \phi\|_{L^\infty} \geq \epsilon_0$$

for some positive constant ϵ_0 . Thanks to this, by Bernstein's inequality, we have

$$\|u_0\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}} \geq \|u_0\|_{\dot{B}_{\infty,\infty}^{-1}} \geq 2^{-j_0} \|\Delta_{j_0} \partial_2 \phi\|_{L^\infty} \geq 2^{-j_0} \epsilon_0,$$

and by triangle inequality

$$\|B_0\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}} \geq \|u_0\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}} - \|u_0 - B_0\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}} \geq 2^{-j_0} \epsilon_0 - C_2 \epsilon^{1-\frac{3}{p}} \geq 2^{-j_0-1} \epsilon_0$$

due to the sufficient small ϵ . Choosing $C_1 = 2^{-j_0} \epsilon_0$ yields the desired result. \square

For some convenience, we provide the following definition of $\chi^s(\mathbb{R}^d)$,

$$\|f\|_{\chi^s} := \int_{\mathbb{R}^d} |\xi|^s |\hat{f}(\xi)| d\xi,$$

and we refer [12] for some details.

Proposition 2.5. *Let $f \in \chi^{-1}$, then we have*

$$\|f\|_{\dot{B}_{\infty,r}^{-1}} \leq \|f\|_{\dot{B}_{\infty,1}^{-1}} \leq \|f\|_{\mathbb{B}_{1,1}^{-1}} \approx \|f\|_{\chi^{-1}},$$

where

$$\|f\|_{\mathbb{B}_{1,1}^{-1}} := \sum_{j \in \mathbb{Z}} 2^{-j} \|\widehat{\Delta_j f}\|_{L^1}.$$

Proof. The first inequality is obvious, while the second inequality can be proved by using $\|f\|_{L^\infty} \leq \|\hat{f}\|_{L^1}$. Now, we prove $\|f\|_{\mathbb{B}_{1,1}^{-1}} \approx \|f\|_{\chi^{-1}}$. By the definition of Δ_j , and using Monotone Convergence Theorem,

$$\begin{aligned} \|f\|_{\mathbb{B}_{1,1}^{-1}} &= \sum_{j \in \mathbb{Z}} 2^{-j} \|\varphi(2^{-j}\xi) \hat{f}(\xi)\|_{L^1} \\ &\approx \sum_{j \in \mathbb{Z}} \| |\xi|^{-1} \varphi(2^{-j}\xi) \hat{f}(\xi) \|_{L^1} \\ &= \| |\xi|^{-1} \hat{f}(\xi) \|_{L^1} \\ &= \|f\|_{\chi^{-1}}, \end{aligned}$$

where we have used $\sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1$ and $\varphi \geq 0$. \square

Lemma 2.6. (i) *Let $(p, r) \in [1, \infty) \times [1, \infty]$, $\operatorname{div} u = 0$, then*

$$\|u \cdot \nabla v\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{\frac{3}{p}-1})} \leq C \left(\|u\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{\frac{3}{p}-1})} \|v\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{\frac{3}{p}+1})} + \|v\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{\frac{3}{p}-1})} \|u\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{\frac{3}{p}+1})} \right); \quad (2.2)$$

(ii) *Let $(p, r) \in [1, \infty) \times [1, \infty]$, $\operatorname{div} u = 0$, then*

$$\|u \cdot \nabla v\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{\frac{3}{p}-1})} \leq C \|v\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{\frac{3}{p}+1})}^{\frac{1+\epsilon}{2}} \|v\|_{\tilde{L}_{t,f}^1(\dot{B}_{p,r}^{\frac{3}{p}-1})}^{\frac{1-\epsilon}{2}}, \quad (2.3)$$

where $0 < \epsilon < 1$ and $f = \|u\|_{\dot{B}_{p,\infty}^{\frac{3}{p}-1}}^{\frac{2}{1-\epsilon}}$. In particular, (2.3) also holds when $(\epsilon, r) = (0, 1)$ and $f = \|u\|_{\dot{B}_{p,1}^{\frac{3}{p}}}^2$.

For the proof, we shall use homogeneous Bony's decomposition:

$$uv = T_u v + T_v u + R(u, v),$$

where

$$T_u v = \sum_{j \in \mathbb{Z}} S_{j-1} u \Delta_j v, \quad T_v u = \sum_{j \in \mathbb{Z}} \Delta_j u S_{j-1} v, \quad R(u, v) = \sum_{j \in \mathbb{Z}} \Delta_j u \tilde{\Delta}_j v,$$

here $\tilde{\Delta}_j = \Delta_{j-1} + \Delta_j + \Delta_{j+1}$.

Proof. The estimate of (2.2) can be established by using

$$\|u \cdot \nabla v\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}} \leq C \{ \|u\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}} \|v\|_{\dot{B}_{p,r}^{\frac{3}{p}+1}} + \|v\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}} \|u\|_{\dot{B}_{p,r}^{\frac{3}{p}+1}} \},$$

whose proof is standard. Thus the goal is the estimate of (2.3). By homogeneous Bony's decomposition,

$$\begin{aligned} \|u \cdot \nabla v\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{\frac{3}{p}-1})} &\leq \|T_{u_i} \partial_i v\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{\frac{3}{p}-1})} + \|T_{\partial_i v} u_i\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{\frac{3}{p}-1})} + \|R(u, \nabla v)\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{\frac{3}{p}-1})} \\ &:= I_1 + I_2 + I_3. \end{aligned} \quad (2.4)$$

Let $\theta = \frac{1-\epsilon}{2}$, $0 < \epsilon < 1$. For I_1 , using Hölder's inequality and Bernstein's inequality,

$$\begin{aligned} I_1 &\leq \left\| 2^{j(\frac{3}{p}-1)} \sum_{|k-j| \leq 4} \|\Delta_j(S_{k-1} u \cdot \nabla \Delta_k v)\|_{L_t^1 L^p} \right\|_{l^r(\mathbb{Z})} \\ &\leq C \left\| 2^{j(\frac{3}{p}-1)} \|S_{j-1} u \cdot \nabla \Delta_j v\|_{L_t^1 L^p} \right\|_{l^r(\mathbb{Z})} \\ &\leq C \left\| 2^{j(\frac{3}{p}-1)} \int_0^t \|S_{j-1} u\|_{L^\infty} \|\nabla \Delta_j v\|_{L^p} d\tau \right\|_{l^r(\mathbb{Z})} \\ &\leq C \left\| 2^{j(\frac{3}{p}+\epsilon)} \int_0^t \|u\|_{\dot{B}_{\infty,\infty}^{-\epsilon}} \|\Delta_j v\|_{L^p} d\tau \right\|_{l^r(\mathbb{Z})} \\ &\leq C \left\| 2^{j(\frac{3}{p}+\epsilon)} \int_0^t \|u\|_{\dot{B}_{p,\infty}^{\frac{3}{p}-\epsilon}} \|\Delta_j v\|_{L^p} d\tau \right\|_{l^r(\mathbb{Z})} \\ &\leq C \left\| 2^{j(\frac{3}{p}+1)(1-\theta)} \|\Delta_j v\|_{L_t^1 L^p}^{1-\theta} \left(\int_0^t 2^{j(\frac{3}{p}-1)} \|u\|_{\dot{B}_{p,\infty}^{\frac{3}{p}-\epsilon}}^{\frac{1}{\theta}} \|\Delta_j v\|_{L^p} d\tau \right)^\theta \right\|_{l^r(\mathbb{Z})} \\ &\leq C \|v\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{\frac{3}{p}+1})}^{1-\theta} \|v\|_{\tilde{L}_{t,f}^1(\dot{B}_{p,r}^{\frac{3}{p}-1})}^\theta = C \|v\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{\frac{3}{p}+1})}^{\frac{1+\epsilon}{2}} \|v\|_{\tilde{L}_{t,f}^1(\dot{B}_{p,r}^{\frac{3}{p}-1})}^{\frac{1-\epsilon}{2}}, \end{aligned}$$

here $f = \|u\|_{\dot{B}_{p,\infty}^{\frac{3}{p}-1}}^{\frac{2}{1-\epsilon}}$ and we have used

$$\|2^{js} S_j u\|_{L^p} \approx \|u\|_{\dot{B}_{p,r}^s}, \quad \forall s < 0.$$

Similarly, for I_2 , by Hölder's inequality and Bernstein's inequality,

$$\begin{aligned}
I_2 &\leq \left\| 2^{j(\frac{3}{p}-1)} \sum_{|k-j|\leq 4} \|\Delta_j(\Delta_k u \cdot \nabla S_{k-1} v)\|_{L_t^1 L^p} \right\|_{l^r(\mathbb{Z})} \\
&\leq C \left\| 2^{j(\frac{3}{p}-1)} \|\Delta_j u \cdot \nabla S_{j-1} v\|_{L_t^1 L^p} \right\|_{l^r(\mathbb{Z})} \\
&\leq C \left\| 2^{j(\frac{3}{p}-1)} \int_0^t \|\Delta_j u\|_{L^p} \|\nabla S_{j-1} v\|_{L^\infty} d\tau \right\|_{l^r(\mathbb{Z})} \\
&\leq C \left\| 2^{j(\epsilon-1)} \int_0^t \|u\|_{\dot{B}_{p,\infty}^{\frac{3}{p}-\epsilon}} \sum_{j'\leq j-2} 2^{j'(\frac{3}{p}+1)} \|\Delta_{j'} v\|_{L^p} d\tau \right\|_{l^r(\mathbb{Z})} \\
&\leq C \left\| \sum_{j'\leq j-2} 2^{(j-j')(\epsilon-1)} \int_0^t \|u\|_{\dot{B}_{p,\infty}^{\frac{3}{p}-\epsilon}} 2^{j'(\frac{3}{p}+\epsilon)} \|\Delta_{j'} v\|_{L^p} d\tau \right\|_{l^r(\mathbb{Z})} \\
&\leq C \left\| 2^{j(\frac{3}{p}+\epsilon)} \int_0^t \|u\|_{\dot{B}_{p,\infty}^{\frac{3}{p}-\epsilon}} \|\Delta_j v\|_{L^p} d\tau \right\|_{l^r(\mathbb{Z})}
\end{aligned}$$

where we have used Young's inequality for series for the last inequality, i.e.,

$$\left\| \sum_{j'\leq j-2} 2^{(j-j')(\epsilon-1)} c_{j'} \right\|_{l^r(\mathbb{Z})} \leq C \|2^{j(\epsilon-1)} \mathbf{1}_{j\geq 2}\|_{l^1(\mathbb{Z})} \|c_j\|_{l^r(\mathbb{Z})} \leq C \|c_j\|_{l^r(\mathbb{Z})}.$$

Following the same argument as I_1 , one gets

$$I_2 \leq C \|v\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{\frac{3}{p}+1})}^{\frac{1+\epsilon}{2}} \|v\|_{\tilde{L}_{t,f}^1(\dot{B}_{p,r}^{\frac{3}{p}-1})}^{\frac{1-\epsilon}{2}}.$$

Finally, we bound I_3 . By Bernstein's inequality, Young's inequality for series and Hölder's inequality, we have

$$\begin{aligned}
I_3 &\leq \left\| 2^{j(\frac{3}{p}-1)} \sum_{k\geq j-3} \|\Delta_j(\Delta_k u \cdot \nabla \tilde{\Delta}_k v)\|_{L_t^1 L^p} \right\|_{l^r(\mathbb{Z})} \\
&\leq C \left\| 2^{j\frac{3}{p}} \sum_{k\geq j-3} \|\Delta_j(\Delta_k u \otimes \tilde{\Delta}_k v)\|_{L_t^1 L^p} \right\|_{l^r(\mathbb{Z})} \\
&\leq C \left\| \sum_{k\geq j-3} 2^{(j-k)\frac{3}{p}} 2^{k\frac{3}{p}} \|\Delta_j(\Delta_k u \otimes \tilde{\Delta}_k v)\|_{L_t^1 L^p} \right\|_{l^r(\mathbb{Z})} \\
&\leq C \left\| 2^{k\frac{3}{p}} \int_0^t \|\Delta_k u\|_{L^p} \|\tilde{\Delta}_k v\|_{L^\infty} d\tau \right\|_{l^r(\mathbb{Z})} \quad (p < \infty) \\
&\leq C \left\| 2^{k(\frac{3}{p}+\epsilon)} \int_0^t \|u\|_{\dot{B}_{p,\infty}^{\frac{3}{p}-\epsilon}} \|\tilde{\Delta}_k v\|_{L^p} d\tau \right\|_{l^r(\mathbb{Z})},
\end{aligned}$$

and using the same way as the estimate of I_1 derives

$$I_3 \leq C \|v\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{\frac{3}{p}+1})}^{\frac{1+\epsilon}{2}} \|v\|_{\tilde{L}_{t,f}^1(\dot{B}_{p,r}^{\frac{3}{p}-1})}^{\frac{1-\epsilon}{2}}.$$

Plugging the above estimates into (2.4) leads the desired result (2.3).

In addition, if $r = 1$, the estimate of I_1 can be replaced as follows:

$$\begin{aligned} I_1 &\leq C \left\| 2^{j(\frac{3}{p}-1)} \int_0^t \|S_{j-1}u\|_{L^\infty} \|\nabla \Delta_j v\|_{L^p} d\tau \right\|_{l^r(\mathbb{Z})} \\ &\leq C \left\| 2^{j\frac{3}{p}} \int_0^t \|u\|_{\dot{B}_{p,1}^{\frac{3}{p}}} \|\Delta_j v\|_{L^p} d\tau \right\|_{l^r(\mathbb{Z})} \\ &\leq C \left\| 2^{j(\frac{3}{p}+1)\frac{1}{2}} \|\Delta_j v\|_{L_t^1 L^p}^{\frac{1}{2}} (2^{j(\frac{3}{p}-1)} \int_0^t \|u\|_{\dot{B}_{p,1}^{\frac{3}{p}}}^2 \|\Delta_j v\|_{L^p} d\tau)^{\frac{1}{2}} \right\|_{l^r} \\ &\leq C \|v\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{\frac{3}{p}+1})}^{\frac{1}{2}} \|v\|_{\tilde{L}_{t,f}^1(\dot{B}_{p,r}^{\frac{3}{p}-1})}^{\frac{1}{2}}, \end{aligned}$$

here $f = \|v\|_{\dot{B}_{p,1}^{\frac{3}{p}}}^2$. At the same time, one can get the new estimates of I_2 and I_3 with the similar procedure. Thus we complete the proof of this lemma. \square

3. PROOF OF THEOREM 1.1

As the Remark 1.7, it suffices to prove the Theorem 1.1 under (1.4). One can get the local existence and uniqueness for (1.1) by using the standard argument on the Navier-Stokes equations, namely, there exists a $T^* > 0$, such that

$$(u, B) \in \tilde{C}([0, T^*]; \dot{B}_{p,r}^{\frac{3}{p}-1}) \cap \tilde{L}^1([0, T^*]; \dot{B}_{p,r}^{\frac{3}{p}+1}).$$

Since the equivalence between (1.1) and (1.2), we will consider (1.2) and suffice to prove $T^* = \infty$.

Now, we begin the proof. Let us consider $0 < \epsilon < 1$ and $r \leq \frac{2}{1-\epsilon}$, containing all cases in (1.6) except $(\epsilon, r) = (0, 1)$. Define

$$\bar{T} := \sup \left\{ t \in (0, T^*) : \|W^-\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{\frac{3}{p}-1})} + \nu_+ \|W^-\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{\frac{3}{p}+1})} \leq \epsilon_0 \nu_+ \right\}, \quad (3.1)$$

where ϵ_0 is small positive constant and will be determined later on.

Step 1. The estimate of W^+ . Consider the first equation in (1.2), using (2.1), we get

$$\frac{d}{dt} \|\Delta_j W^+\|_{L^p} + c\nu_+ 2^{2j} \|\Delta_j W^+\|_{L^p} \leq C \|\Delta_j (W^- \cdot \nabla W^+)\|_{L^p} + C\nu_- 2^{2j} \|\Delta_j W^-\|_{L^p},$$

which yields by a standard procedure

$$\begin{aligned} & \|W^+\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{\frac{3}{p}-1})} + c\nu_+ \|W^+\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{\frac{3}{p}+1})} \\ & \leq 2\|W_0^+\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}} + C\|W^- \cdot \nabla W^+\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{\frac{3}{p}-1})} + C\nu_- \|W^-\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{\frac{3}{p}+1})}. \end{aligned}$$

By (2.2) and (3.1), we have for all $t \in (0, \bar{T}]$,

$$\begin{aligned} & \|W^+\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{\frac{3}{p}-1})} + c\nu_+ \|W^+\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{\frac{3}{p}+1})} \leq 2\|W_0^+\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}} + C\nu_- \|W^-\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{\frac{3}{p}+1})} \\ & \quad + C(\|W^-\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{\frac{3}{p}-1})} \|W^+\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{\frac{3}{p}+1})} + \|W^+\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{\frac{3}{p}-1})} \|W^-\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{\frac{3}{p}+1})}) \\ & \leq 2\|W_0^+\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}} + C\epsilon_0(\nu_+ \|W^+\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{\frac{3}{p}+1})} + \|W^+\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{\frac{3}{p}-1})}) + C\epsilon_0\nu_-, \end{aligned}$$

with the selection of $\epsilon_0 < \min\{\frac{c}{2C}, \frac{1}{2C}\}$ leads

$$\|W^+\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{\frac{3}{p}-1})} + c\nu_+ \|W^+\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{\frac{3}{p}+1})} \leq 4\|W_0^+\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}} + 2c\nu_-. \quad (3.2)$$

Step 2. The estimate of W^- . Denote

$$f(t) := \|W^+(t)\|_{\dot{B}_{p,\infty}^{\frac{2}{1-\epsilon}}}^{\frac{2}{1-\epsilon}}, \quad W_\lambda^\pm := W^\pm \exp\{-\lambda \int_0^t f(\tau) d\tau\}, \quad p_\lambda := p \exp\{-\lambda \int_0^t f(\tau) d\tau\},$$

where λ is large enough constant and will be determined later on. So we can rewrite the second equation in (1.2) as

$$\partial_t W_\lambda^- + \lambda f(t) W_\lambda^- + W^+ \cdot \nabla W_\lambda^- + \nabla p_\lambda - \nu_+ \Delta W_\lambda^- = \nu_- \Delta W_\lambda^+.$$

By a similar procedure, we have

$$\begin{aligned} & \|\Delta_j W_\lambda^-\|_{L_t^\infty L^p} + \lambda \int_0^t f(\tau) \|\Delta_j W_\lambda^-\|_{L^p} d\tau + c\nu_+ 2^{2j} \|\Delta_j W_\lambda^-\|_{L_t^1 L^p} \\ & \leq \|\Delta_j W_0^-\|_{L^p} + C\|\Delta_j(W^+ \cdot \nabla W_\lambda^-)\|_{L_t^1 L^p} + C\nu_- 2^{2j} \|\Delta_j W_\lambda^+\|_{L_t^1 L^p}. \end{aligned}$$

Then we obtain

$$\begin{aligned} & \|W_\lambda^-\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{\frac{3}{p}-1})} + c\nu_+ \|W_\lambda^-\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{\frac{3}{p}+1})} + \lambda \|W_\lambda^-\|_{\tilde{L}_{t,f}^1(\dot{B}_{p,r}^{\frac{3}{p}-1})} \\ & \leq \|W_0^-\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}} + C\nu_- \|W_\lambda^+\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{\frac{3}{p}+1})} + C\|W^+ \cdot \nabla W_\lambda^-\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{\frac{3}{p}-1})}. \end{aligned}$$

Thanks to (2.3), and by Young's inequality, we obtain

$$\begin{aligned} & \|W_\lambda^-\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{\frac{3}{p}-1})} + c\nu_+ \|W_\lambda^-\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{\frac{3}{p}+1})} + \lambda \|W_\lambda^-\|_{\tilde{L}_{t,f}^1(\dot{B}_{p,r}^{\frac{3}{p}-1})} \\ & \leq \|W_0^-\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}} + C\nu_- \|W_\lambda^+\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{\frac{3}{p}+1})} + C\|W_\lambda^-\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{\frac{3}{p}+1})}^{\frac{1+\epsilon}{2}} \|W_\lambda^-\|_{\tilde{L}_{t,f}^1(\dot{B}_{p,r}^{\frac{3}{p}-1})}^{\frac{1-\epsilon}{2}} \\ & \leq \|W_0^-\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}} + C\nu_- \|W_\lambda^+\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{\frac{3}{p}+1})} + \frac{c\nu_+}{8} \|W_\lambda^-\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{\frac{3}{p}+1})} \\ & \quad + C\nu_+^{-\frac{1+\epsilon}{1-\epsilon}} \|W_\lambda^-\|_{\tilde{L}_{t,f}^1(\dot{B}_{p,r}^{\frac{3}{p}-1})}. \end{aligned} \quad (3.3)$$

Choosing $\lambda > 2C\nu_+^{-\frac{1+\epsilon}{1-\epsilon}}$, absorbing the third and fourth term on the right hand side of last inequality by the left hand side in (3.3) follows

$$\begin{aligned} & \|W_\lambda^-\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{\frac{3}{p}-1})} + \frac{7c}{8}\nu_+\|W_\lambda^-\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{\frac{3}{p}+1})} + C\nu_+^{-\frac{1+\epsilon}{1-\epsilon}}\|W_\lambda^-\|_{\tilde{L}_{t,f}^1(\dot{B}_{p,r}^{\frac{3}{p}-1})} \\ & \leq \|W_0^-\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}} + C\nu_-\|W_\lambda^+\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{\frac{3}{p}+1})}. \end{aligned}$$

Obviously, using (3.2), we have

$$\begin{aligned} & \|W_\lambda^-\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{\frac{3}{p}-1})} + c\nu_+\|W_\lambda^-\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{\frac{3}{p}+1})} \leq 2\|W_0^-\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}} + C\nu_-\|W_\lambda^+\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{\frac{3}{p}+1})} \\ & \leq C \left(\|W_0^-\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}} + \frac{\nu_-}{\nu_+}(\|W_0^+\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}} + \nu_-) \right). \end{aligned}$$

This yields, after using (3.2) again, for all $t \in (0, \bar{T})$,

$$\begin{aligned} & \|W_\lambda^-\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{\frac{3}{p}-1})} + c\nu_+\|W_\lambda^-\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{\frac{3}{p}+1})} \\ & \leq C \left(\|W_0^-\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}} + \frac{\nu_-}{\nu_+}(\|W_0^+\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}} + \nu_-) \right) \exp \left\{ C\nu_+^{-\frac{1+\epsilon}{1-\epsilon}} \int_0^t \|W^+(\tau)\|_{\dot{B}_{p,\infty}^{\frac{3}{p}-\epsilon}}^{\frac{2}{1-\epsilon}} d\tau \right\} \\ & \leq C \left(\|W_0^-\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}} + \frac{\nu_-}{\nu_+}(\|W_0^+\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}} + \nu_-) \right) \exp \left\{ C\nu_+^{-\frac{1+\epsilon}{1-\epsilon}} \|W^+\|_{\tilde{L}_t^{\frac{2}{1-\epsilon}}(\dot{B}_{p,r}^{\frac{3}{p}-\epsilon})}^{\frac{2}{1-\epsilon}} \right\} \\ & \leq C \left(\|W_0^-\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}} + \frac{\nu_-}{\nu_+}(\|W_0^+\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}} + \nu_-) \right) \exp \left\{ C\nu_+^{-\frac{2}{1-\epsilon}} (\nu_- + \|W_0^+\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}})^{\frac{2}{1-\epsilon}} \right\}, \end{aligned}$$

which implies that if we take η small enough in (1.4), there holds for all $t \leq \bar{T}$,

$$\|W_\lambda^-\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{\frac{3}{p}-1})} + \nu_+\|W_\lambda^-\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{\frac{3}{p}+1})} \leq C\eta\nu_+ < \frac{\epsilon_0}{2}\nu_+.$$

Then by a standard continuous method, we get $\bar{T} = T^* = \infty$.

The remainder is $r = 1, \epsilon = 0$, by a similar arguments, using (2.3) for this case and let $(\epsilon, r) = (0, 1)$, $f = \|W^+\|_{\dot{B}_{p,1}^{\frac{3}{p}}}$ in (3.3), the desired result can be obtained. Hence, we complete the proof of Theorem 1.1.

4. PROOF OF THEOREM 1.4

One can easily get the local well-posedness of (1.1), that is, there exists a $T^* > 0$ such that

$$(u, B) \in C([0, T^*]; \chi^{-1}(\mathbb{R}^3)) \cap L^1([0, T^*]; \chi^1(\mathbb{R}^3)).$$

So we suffices to show $T^* = \infty$.

Now, we begin the proof. (1.10) is indeed equal to

$$\left(\|W_0^-\|_{\chi^{-1}} + \frac{C\nu_-}{\nu_+}(\nu_- + \|W_0^+\|_{\chi^{-1}}) \right) \exp \left\{ \frac{C}{\nu_+^2}(\nu_- + \|W_0^+\|_{\chi^{-1}})^2 \right\} \leq (2 - \epsilon_0)\nu_+ \quad (4.1)$$

for some $\epsilon_0 > 0$. And next we suffices to prove the desired result under (4.1). Let $C_1, C_2 \in (0, 2)$ satisfying $2(2 - \epsilon_0)^2 < C_2(2 - C_1)^2$ and

$$a = C_2\nu_+, \quad a_1 = C_1\nu_+, \quad b \in \left(\frac{2(2 - \epsilon_0)}{2 - C_1}\nu_+, \sqrt{2C_2}\nu_+\right).$$

Then consider the first equation in (1.2), by the procedure as [12], and using interpolation inequality, we have

$$\begin{aligned} \frac{d}{dt} \|W^+\|_{\chi^{-1}} + \nu_+ \|W^+\|_{\chi^1} &\leq \|W^+ \cdot \nabla W^-\|_{\chi^{-1}} + \nu_- \|W^-\|_{\chi^1} \\ &\leq \|W^+\|_{\chi^0} \|W^-\|_{\chi^0} + \nu_- \|W^-\|_{\chi^1} \\ &\leq \|W^-\|_{\chi^0}^2 \|W^+\|_{\chi^{-1}}^{\frac{1}{2}} \|W^+\|_{\chi^1}^{\frac{1}{2}} + \nu_- \|W^-\|_{\chi^1} \\ &\leq \frac{1}{2a} \|W^-\|_{\chi^0}^2 \|W^+\|_{\chi^{-1}} + \frac{a}{2} \|W^+\|_{\chi^1} + \nu_- \|W^-\|_{\chi^1}, \end{aligned}$$

which derives by integrating in time,

$$\begin{aligned} &\|W^+\|_{L_t^\infty(\chi^{-1})} + (\nu_+ - \frac{a}{2}) \|W^+\|_{L_t^1(\chi^1)} \\ &\leq \frac{1}{2a} \|W^-\|_{L_t^2(\chi^0)}^2 \|W^+\|_{L_t^\infty(\chi^{-1})} + \nu_- \|W^-\|_{L_t^1(\chi^1)} + \|W_0^+\|_{\chi^{-1}}. \end{aligned} \quad (4.2)$$

Define

$$\bar{T} := \sup \{t \in (0, T^*) : \|W^-\|_{L_t^\infty(\chi^{-1})} + \nu_+ \|W^-\|_{L_t^1(\chi^1)} \leq b\}. \quad (4.3)$$

Then we will prove $T^* = \bar{T} = \infty$ under (4.1). Using (4.3), combining with (4.2), we have

$$(1 - \frac{b^2}{2a\nu_+}) \|W^+\|_{L_t^\infty(\chi^{-1})} + (\nu_+ - \frac{a}{2}) \|W^+\|_{L_t^1(\chi^1)} \leq \|W_0^+\|_{\chi^{-1}} + \frac{b\nu_-}{\nu_+}. \quad (4.4)$$

Following the similar way as (4.2), one gets

$$\frac{d}{dt} \|W^-\|_{\chi^{-1}} + \nu_+ \|W^-\|_{\chi^1} \leq \frac{1}{2a_1} \|W^+\|_{\chi^0}^2 \|W^-\|_{\chi^{-1}} + \frac{a_1}{2} \|W^-\|_{\chi^1} + \nu_- \|W^+\|_{\chi^1}$$

and thanks to (4.4),

$$\begin{aligned} &\|W^-(t)\|_{\chi^{-1}} + (\nu_+ - \frac{a_1}{2}) \|W^-\|_{L_t^1(\chi^1)} \\ &\leq \frac{1}{2a_1} \int_0^t \|W^+\|_{\chi^0}^2 \|W^-\|_{\chi^{-1}} d\tau + \frac{\nu_-}{\nu_+ - \frac{a}{2}} (\frac{b\nu_-}{\nu_+} + \|W_0^+\|_{\chi^{-1}}) + \|W_0^-\|_{\chi^{-1}}, \end{aligned}$$

with the application of Gronwall's lemma, by interpolation's inequality and (4.4) leads

$$\begin{aligned} &\|W^-\|_{L_t^\infty(\chi^{-1})} + (\nu_+ - \frac{a_1}{2}) \|W^-\|_{L_t^1(\chi^1)} \\ &\leq (\|W_0^-\|_{\chi^{-1}} + \frac{\nu_-}{\nu_+ - \frac{a}{2}} (\frac{b\nu_-}{\nu_+} + \|W_0^+\|_{\chi^{-1}})) \exp \left\{ \frac{1}{2a_1} \int_0^t \|W^+\|_{\chi^0}^2 d\tau \right\} \\ &\leq (\|W_0^-\|_{\chi^{-1}} + \frac{\nu_-}{\nu_+ - \frac{a}{2}} (\frac{b\nu_-}{\nu_+} + \|W_0^+\|_{\chi^{-1}})) \exp \left\{ \frac{1}{2a_1} \|W^+\|_{L_t^\infty(\chi^{-1})} \|W^+\|_{L_t^1(\chi^1)} \right\} \\ &\leq (\|W_0^-\|_{\chi^{-1}} + \frac{\nu_-}{\nu_+ - \frac{a}{2}} (\frac{b\nu_-}{\nu_+} + \|W_0^+\|_{\chi^{-1}})) \exp \left\{ \frac{2a\nu_+}{a_1(2a\nu_+ - b^2)(2\nu_+ - a_1)} (\frac{b\nu_-}{\nu_+} + \|W_0^+\|_{\chi^{-1}})^2 \right\}. \end{aligned}$$

which indicates that there exists constant C such that

$$\begin{aligned} & \|W^-\|_{L_t^\infty(\chi^{-1})} + (1 - \frac{C_1}{2})\nu_+ \|W^-\|_{L_t^1(\chi^1)} \\ & \leq \left(\|W_0^-\|_{\chi^{-1}} + \frac{C\nu_-}{\nu_+}(\nu_- + \|W_0^+\|_{\chi^{-1}}) \right) \exp \left\{ \frac{C}{\nu_+^2}(\nu_- + \|W_0^+\|_{\chi^{-1}})^2 \right\} \\ & \leq (2 - \epsilon_0)\nu_+. \end{aligned}$$

This implies that

$$\|W^-\|_{L_t^\infty(\chi^{-1})} + \nu_+ \|W^-\|_{L_t^1(\chi^1)} < \frac{2(2 - \epsilon_0)}{2 - C_1} < b.$$

Therefore, by standard continuous method, we get $T^* = \bar{T} = \infty$. This concludes the proof of Theorem 1.4.

REFERENCES

- [1] H. Bahouri, J.-Y. Chemin, R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Grundlehren der mathematischen Wissenschaften, Springer, Heidelberg, 2011.
- [2] J. Bourgain and N. Pavlovic, Ill-posedness for the Navier-Stokes equations in a critical Besov space in 3D, *J. Funct. Anal.* **255**, (2008), 2233-2247.
- [3] J.-Y. Chemin, I. Gallagher, Wellposedness and stability results for the Navier-Stokes equations in \mathbb{R}^3 , *Ann. Inst. H. Poincaré Anal. Non Linéaire* **26**, (2009), 599-624.
- [4] J.-Y. Chemin, D.S. McCormick, J.C. Robinson, J.L. Rodrigo, Local existence for the non-resistive MHD equations in Besov space, arXiv:1503.01651v1 [math.AP] 5 Mar 2015.
- [5] Q. Chen, C. Miao and Z. Zhnag, On the regularity criterion of weak solution for the 3D viscous magneto-hydrodynamics equations. *Comm. Math. Phys.* **284**, (2008), 919-930.
- [6] R. Danchin, Local theory in critical spaces for compressible viscous and heat-conductive gases, *Comm. Partial Differential Equations* **26**, (2001), 1183-1233.
- [7] G. Duraut and J.L. Lions, Inégalité en thermoélasticité et magnéto-hydrodynamic equations, *Arch. Ration. Mech. Anal.* **46** (1972), 241-247.
- [8] C.L. Fefferman, D.S. McCormick, J.C. Robinson, J.L. Rodrigo, Higher order commutator estimates and local existence for the non-resistive MHD equations and related models, *J. Funct. Anal.* **267** (2014), 1035-1056.
- [9] L. Grafakos, *Modern Fourier Analysis*. 2nd Edition., Grad. Text in Math., **250**, Springer-Verlag, 2008.
- [10] C. He, X. Huang and Y. Wang, On some new global existence results for 3D magnetohydrodynamic equations, *Nonlinearity* **27**, (2014), 343-352.
- [11] C. He and Z.P. Xin, On the regularity of solutions to magnetohydrodynamic equations, *J. Diff. Eqns.* **213**, (2005), 235-254.
- [12] Z. Lei and F.H. Lin, Global mild solutions of Navier-Stokes equations, *Comm. Pure Appl. Math.* **64**, (2011), 1297-1304.
- [13] F.H. Lin and P. Zhang, Global small solutions to an MHD-type system: the three-dimensional case, *Comm. Pure Appl. Math.* **67**, (2014), 531-580.
- [14] F.H. Lin and T. Zhang, Global small solutions to a complex fluid model in three dimensional, *Arch. Ration. Mech. Anal.* **216**, (2015), 905-920.
- [15] X. Ren, J. Wu, Z. Xiang and Z. Zhang, Global existence and decay of smooth solution for the 2-D MHD equations without magnetic diffusion, *J. Funct. Anal.* **267**, (2014), 503-541.
- [16] M. Sermenge and R. Termam, Some mathematical questions related to the MHD equations, *Comm. Pure Appl. Math.* **46** (1983), 635-664.
- [17] E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, 1970.

- [18] R. Wan, On the uniqueness for the 2D MHD equations without magnetic diffusion, arXiv:1503.03589v1 [math.AP] 12 Mar 2015.
- [19] B. Wang, Ill-posedness for the Navier-Stokes equations in critical Besov spaces $\dot{B}_{\infty,q}^{-1}$, *Adv. Math.* **268**, (2015), 350-372.
- [20] Y. Wang and K. Wang, Global well-posedness of the three dimensional magnetohydrodynamics equations, *Nonlinear Anal. Real World Appl.* **17**, (2014), 245-251.
- [21] J. Wu, Analytic results related to magneto-hydrodynamic turbulence, *Physica D* **136**, (2000), 353-372.
- [22] J. Wu, Bounds and new approaches for the 3D MHD equations, *J. Nonlinear Sci* **12**, (2002), 395-413.
- [23] J. Wu, Regularity results for weak solutions of the 3D MHD equations, *Discrete Contin. Dyn. Syst.* **10**, (2004), 543-556.
- [24] L. Xu and P. Zhnag, Global small solutions to three-dimensional incompressible magnetohydrodynamical system, *SIAM J. Math. Anal.* **47**, (2015), 26-65.
- [25] Y. Zhou, Remarks on the regularities for the 3D MHD equations, *Discrete Contin. Dyn. Syst.* **12**, (2005), 881-886.
- [26] Y. Zhou, Regularity criteria for the generalized viscous MHD equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **24**, (2007), 491-505.

¹ SCHOOL OF MATHEMATICS SCIENCES, ZHEJIANG UNIVERSITY, HANZHOU 310027, CHINA
E-mail address: rhwanmath@zju.edu.cn, rhwanmath@163.com